

# Approximate eigenvalue and eigenfunction solutions for the generalized Hulthén potential with any angular momentum

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An approximate solution of the Schrödinger equation for the generalized Hulthén potential with non-zero angular quantum number is solved. The bound state energy eigenvalues and eigenfunctions are obtained in terms of Jacobi polynomials. The Nikiforov–Uvarov method is used in the computations. We have considered the time-independent Schrödinger equation with the associated form of Hulthén potential which simulate the effect of the centrifugal barrier for any  $l$ -state. The energy levels of the used Hulthén potential gives satisfactory values for the non-zero angular momentum as the generalized Hulthén effective potential.

**KEY WORDS:** energy eigenvalues and eigenfunctions, Hulthén potential, Nikiforov–Uvarov method

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## 1. Introduction

The Hulthén potential [1] is a short-range potential in physics. The potential form is

$$V^{(H)}(r) = -Ze^2\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}}, \quad (1)$$

where  $Z$  is a constant and  $\delta$  is the range (screening) parameter. If the potential is used for atomic nuclei, the  $Z$  is identified with the atomic number. This potential

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is a special case of the Eckart potential [2], which has been widely used in several branches of physics and its bound states and scattering properties have been investigated by a variety of techniques [3].

The radial Schrödinger equation for the Hulthén potential can be solved analytically only for  $s$ -states ( $l = 0$ ) [1,4,5]. For  $l \neq 0$ , a number of methods have been used to find bound-state energies numerically [3,6–16] and analytically [17,18]. In the context, we present briefly Nikiforov–Uvarov (NU) method [19] in the frame of quantum mechanics [20–23] and solve analytically the generalized Hulthén potential for non-zero angular momentum states given by Gönül, et al. [18], Greene and Aldrich [24], and Simsek and Egrifes [25]:

$$V_{\text{eff}}^{(H)}(r) = V_{(l+1)}(r) = -Ze^2\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} + \frac{l(l+1)\hbar^2\delta^2}{2m} \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2}, \quad (2)$$

which is for  $s$ -states is not shape invariant [26] and leads to the usual Hulthén potential in equation (1). On the other hand, equation (2) is well-known as the approximate Hulthén effective potential introduced by Greene and Aldrich [24].<sup>1</sup> It is of much concern to see that for small values of  $\delta r$ , equation (2) approximates to the effective exponential-type screened Coulomb potential given by

$$V_{\text{eff}}^{(H)}(r, \delta \approx 0) \rightarrow V_{\text{eff}}^{(\text{SC})} = -Ze^2 \frac{e^{-\delta r}}{r} + \frac{l(l+1)\hbar^2}{2mr^2}. \quad (3)$$

In the case of Coulomb potential, the Hamiltonian corresponds to the addition of an appropriate barrier potential and the so-called degeneracy is recovered as a natural consequence [18,25]. At small values of  $r$ , the Hulthén potential, behaves like a screened Coulomb potential whereas for large values of  $r$  it decreases exponentially so its capacity for bound state is smaller than the Coulomb potential. The coulomb problem is analytically solvable for all energies and angular momenta. Because of the similarity and points of contrast mentioned above between Coulomb and Hulthén potentials, it may be of much interest to use the generalized Hulthén potential, equation (2), to generate their eigenvalues and eigenfunctions by applying NU method. The outline of the paper is as follows: in section 2, we solve the Schrödinger equation (SE) with Hulthén superpotential part for its energy eigenvalues and eigenfunctions. In section 3, we consider the  $l = 0$  and  $l \neq 0$  cases and compare with the other works and methods. Finally, section 4 is for our conclusions.

<sup>1</sup>In [27] the Hulthén effective potential is taken as  $V_{\text{eff}}^{(H)}(r) = -\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} + \frac{l(l+1)}{2} \left( \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} \right)^2$  in atomic units.

## 2. Polynomial solution of the Hulthén potential

The NU method provides us an exact solution of non-relativistic SE for certain kind of potentials [19]. The method is based upon the solutions of general second-order linear differential equation with special orthogonal functions [28]. For a given potential, the SE is reduced to a generalized equation of hypergeometric type with an appropriate  $s = s(r)$  coordinate transformation. Thus, it takes the form:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{4}$$

where  $\sigma(s)$  and  $\tilde{\sigma}(s)$  are polynomials of degree at most 2, and  $\tilde{\tau}(s)$  a polynomial of at most 1. Using the transformation

$$\psi(s) = \Phi(s)y(s), \tag{5}$$

equation (4) could be reduced to the type equation

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \tag{6}$$

where  $\Phi(s)$  is found to satisfy the condition  $\Phi'(s)/\Phi(s) = \pi(s)/\sigma(s)$ . Further,  $y(s)$  is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation

$$y(s, \lambda_n) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \tag{7}$$

where  $B_n$  is a normalizing constant and the weight function  $\rho(s)$  satisfies the equation [19]

$$\frac{d}{ds} (\sigma(s)\rho(s)) = \tau(s)\rho(s). \tag{8}$$

The function  $\pi(s)$  and the parameter  $\lambda$  required for this method are defined by

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{9}$$

and

$$\lambda = k + \pi'(s). \tag{10}$$

Here,  $\pi(s)$  is a polynomial with the parameter  $s$  and the determination of  $k$  is the essential point in the calculation of  $\pi(s)$ . Thus, for the determination of  $k$ ,

the discriminant under the square root is being set equal to zero and the resulting the polynomial of degree 2, which has to be solved for its roots  $k_{1,2}$ . Hence, a new eigenvalue equation for the SE becomes

$$\lambda_n + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) = 0, \quad n = 0, 1, 2, \dots, \tag{11}$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{12}$$

and it will have a negative derivative.

In order to apply the NU method, we have to write SE for the generalized Hulthén (Hulthén potential-plus-its square potentials) as

$$\left\{ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left[ E + \frac{V_1 e^{-\delta r}}{1 - e^{-\delta r}} - \frac{V_2 e^{-2\delta r}}{(1 - e^{-\delta r})^2} \right] \right\} R(r) = 0 \tag{13}$$

with coupling constants

$$V_1 = Ze^2\delta \left[ 1 - l(l+1) \frac{\hbar^2\delta}{2mZe^2} \right], \quad \text{and} \quad V_2 = \frac{\hbar^2\delta^2}{2m} l(l+1). \tag{14}$$

The aim of using the square of the Hulthén potentials in equation (13) is to substitute the centrifugal part in the SE. Thus, using the assignment  $s = e^{-\delta r}$ , then equation (13) becomes

$$\frac{d^2 R(s)}{ds^2} + \frac{1}{s} \frac{dR(s)}{ds} + \frac{2m}{\hbar^2\delta^2 s^2} \left[ E + \frac{V_1 s}{1-s} - \frac{V_2 s^2}{(1-s)^2} \right] R(s) = 0 \tag{15}$$

and also introducing the given dimensionless parameters

$$\epsilon = -\frac{2mE}{\hbar^2\delta^2} > 0, \quad (E < 0), \quad \beta = \frac{2mV_1}{\hbar^2\delta^2}, \quad (\beta > 0), \quad \gamma = \frac{2mV_2}{\hbar^2\delta^2}, \quad (\gamma > 0) \tag{16}$$

finally it leads into the following simple hypergeometric form given by

$$\frac{d^2 R(s)}{ds^2} + \frac{(1-s)}{s(1-s)} \frac{dR(s)}{ds} + \frac{1}{[s(1-s)]^2} \times \left[ -(\epsilon + \beta + \gamma)s^2 + (2\epsilon + \beta)s - \epsilon \right] R(s) = 0. \tag{17}$$

Hence, comparing the last equation with the generalized hypergeometric type, equation (4), we obtain the associated polynomials as

$$\tilde{\tau}(s) = 1 - s, \quad \sigma(s) = s(1 - s), \quad \tilde{\sigma}(s) = -(\epsilon + \beta + \gamma)s^2 + (2\epsilon + \beta)s - \epsilon. \tag{18}$$

When these polynomials are substituted into equation (9), with  $\sigma'(s) = 1 - 2s$ , we obtain

$$\pi(s) = -\frac{s}{2} \pm \frac{1}{2} \sqrt{(1 + 4\epsilon + 4\beta + 4\gamma + 4k) s^2 - 4(\beta + 2\epsilon + k) s + 4\epsilon}. \tag{19}$$

Further, the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, it becomes

$$\Delta = [4(\beta + 2\epsilon + k)]^2 - 4 \times 4\epsilon(1 + 4\epsilon + 4\beta + 4\gamma + 4k) = 0. \tag{20}$$

Solving equation (20) for the constant  $k$ , we get the double roots as  $k_{+,-} = -\beta \pm \sqrt{\epsilon(1 + 4\gamma)}$ , and substituting these values for each  $k$  into equation (19), we obtain

$$\pi(s) = -\frac{s}{2} \pm \frac{1}{2} \begin{cases} [(2\sqrt{\epsilon} - \sqrt{1 + 4\gamma})s - 2\sqrt{\epsilon}] & \text{for } k_+ = -\beta + \sqrt{\epsilon(1 + 4\gamma)}, \\ [(2\sqrt{\epsilon} + \sqrt{1 + 4\gamma})s - 2\sqrt{\epsilon}] & \text{for } k_- = -\beta - \sqrt{\epsilon(1 + 4\gamma)}. \end{cases} \tag{21}$$

Making the following choice for the polynomial  $\pi(s)$  as

$$\pi(s) = -\frac{s}{2} - \frac{1}{2} [(2\sqrt{\epsilon} + \sqrt{1 + 4\gamma})s - 2\sqrt{\epsilon}] \tag{22}$$

gives the function:

$$\tau(s) = 1 - 2s - [(2\sqrt{\epsilon} + \sqrt{1 + 4\gamma})s - 2\sqrt{\epsilon}], \tag{23}$$

which has a negative derivative of the form  $\tau'(s) = -(2 + 2\sqrt{\epsilon} + \sqrt{1 + 4\gamma})$ . Thus, from equations (10) and (11), we find

$$\lambda = -\beta - \frac{1}{2}(1 + 2\sqrt{\epsilon})(1 + \sqrt{1 + 4\gamma}) \tag{24}$$

and

$$\lambda_n = -(2 + 2\sqrt{\epsilon} + \sqrt{1 + 4\gamma})n - n(n - 1). \tag{25}$$

After setting  $\lambda_n = \lambda$  and solving for  $\epsilon$ , we find:

$$\epsilon_n = \left[ \frac{1 + 2n}{2} - \frac{(n(n + 1) + \beta)}{1 + 2n + \sqrt{1 + 4\gamma}} \right]^2, \tag{26}$$

which is exactly equation (26) in [29] for the generalized Woods–Saxon potential if one lets  $q = -1$  and  $a = 1/\delta$ . Therefore, substituting the values of  $\epsilon$ ,  $\beta$ , and

$\gamma$  into equation (26), one can immediately determine the Hulthén's exact energy eigenvalues  $E_{n,l+1}$  as

$$E_{n,l+1}^{(H)} = -\frac{\hbar^2 \delta^2}{2m} \left[ \frac{1+2n}{2} - \frac{\left( n(n+1) + \frac{2mV_1}{\hbar^2 \delta^2} \right)}{1+2n + \sqrt{1 + \frac{8mV_2}{\hbar^2 \delta^2}}} \right]^2, \quad 0 \leq n < \infty. \tag{27}$$

Therefore, substituting, equation (14) into equation (27), one gets

$$E_{n,l+1}^{(H)} = -\frac{\hbar^2}{2m} \left[ \frac{(me^2 Z / \hbar^2)}{n+l+1} - \frac{(n+l+1)}{2} \delta \right]^2, \quad 0 \leq n < \infty \tag{28}$$

for  $l+1$  Hulthén superpotential. Following Ref. [29], in atomic units ( $\hbar = m = c = e = 1$ ) and for  $Z = 1$ , equation (28) turns out to be

$$E_{n,l+1}^{(H)} = -\frac{1}{2} \left[ \frac{1}{n+l+1} - \frac{(n+l+1)}{2} \delta \right]^2, \quad 0 \leq n < \infty, \quad l = 0, 1, 2, \dots, \tag{29}$$

or

$$E_{\bar{n},l}^{(H)} = -\frac{1}{2} \left[ \frac{1}{\bar{n}+l} - \frac{\bar{n}+l}{2} \delta \right]^2, \quad \bar{n} = n+1, \quad l = 0, 1, 2, \dots, \tag{30}$$

which is exactly the same result obtained by other works (cf. e.g., Ref. [29], equation (78)) if  $l$  is set equal to zero. The above-equation indicates that we deal with a family of Hulthén potentials.<sup>2</sup> Equation (28) agrees with equation (5) in [18] for  $l \neq 0$  case, and equation (11) in [27] for  $l = 0$  case. Of course, it is clear that by imposing appropriate changes in the parameters  $\delta$  and  $V_1$ , the index  $n$  describes the quantization for the bound energy states. In addition, if the parameter  $V_2$  in equation (27) is adjusted to zero, solution reduces to the form obtained for the standard Hulthén potential without a barrier term.

Let us now find the corresponding wavefunctions. Applying the NU method, the polynomial solutions of the hypergeometric function  $y(s)$  depends on the determination of weight function  $\rho(s)$  which is found to be

$$\rho(s) = (1-s)^{\eta-1} s^{2\sqrt{\epsilon}}; \quad \eta = 1 + \sqrt{1+4\gamma}. \tag{31}$$

Substituting into the Rodrigues relation given in equation (8), the eigenfunctions are obtained in the following form

$$y_{n,q}(s) = C_n (1-s)^{-(\eta-1)} s^{-2\sqrt{\epsilon}} \frac{d^n}{ds^n} \left[ (1-s)^{n+\eta-1} s^{n+2\sqrt{\epsilon}} \right], \tag{32}$$

where  $C_n$  stands for the normalization constant and its value is  $1/n!$ . The polynomial solutions of  $y_n(s)$  are expressed in terms of Jacobi Polynomials, which

<sup>2</sup>The critical screening  $\delta_c$ , at which  $E_n = 0$ , is defined, in atomic units, by  $\delta_c = 2/(n+l+1)^2$ .

are of the classical orthogonal polynomials, with weight function  $(1 - s)^{\eta-1} s^{2\sqrt{\epsilon}}$  in the closed interval  $[0, 1]$ , yielding  $A_n P_n^{(2\sqrt{\epsilon}, \eta-1)}(1 - 2s)$  [28]. Finally, the other part of the wave function in equation (5) is found to be

$$\Phi(s) = (1 - s)^\mu s^{\sqrt{\epsilon}}, \quad \mu = \eta/2. \tag{33}$$

Combining the Jacobi polynomials and  $\Phi(s)$  in equation (33), the  $s$ -wave functions ( $l = 0$ ) could be determined as

$$R_n(s) = D_n s^{\sqrt{\epsilon}} (1 - s)^\mu P_n^{(2\sqrt{\epsilon}, \eta-1)}(1 - 2s) \tag{34}$$

with  $s = e^{-\delta r}$  and  $D_n$  is a new normalization constant.

### 3. Conclusions

The radial SE is solved for the Hulthén potential with any angular momentum. The NU method has been used. Further, we have computed the eigenvalues and eigenfunctions from the real form of this potential. Therefore, it is found that the energy eigenvalues are consistent with the results obtained by using other methods. In this regard, figure 1 shows the variation of the Hulthén potential

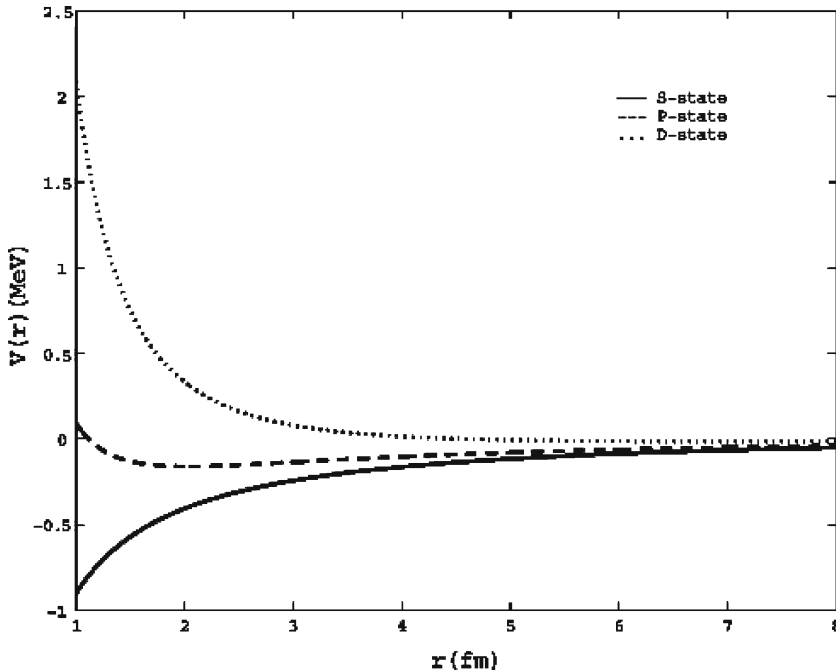


Figure 1. Variation of the Hulthén potential as a function  $r$ . The curves are shown for screening parameter  $\delta = 0.2$  for the  $s$ -,  $p$ -, and  $d$ -states.

with  $r$  for  $s$ -,  $p$ -, and  $d$ -states. Figure 2 plots the variation of the Hulthén potential with  $r$  for the  $s$ -state with various screening parameters  $\delta = 0.002, 0.01$ , and  $0.1$ . Figures 3 and 4 show the variation of the energy eigenvalues with respect to the quantum number  $n$  for  $s$ - and  $p$ -states with the a chosen values of screening parameter  $\delta = 0.002, 0.05$ , and  $0.2$ , respectively. Moreover, table 1 shows the bound energy eigenvalues of the Hulthén potential as a function of  $\delta$  for various quantum numbers of  $S$ -state. These results are compared with other works [11, 30]. Further, the bound energy eigenvalues as a function of  $\delta$  for the states  $2p$  ( $n = 0, l = 1$ ) and  $3d$  ( $n = 1, l = 1$ ) are given in table 2. Comparison of our results with numerical data of [3, 27] is also given. Here the small difference in the energy is due to a difference in the forms of the potentials used (cf. Ref. [27]). Better results have been obtained for the  $2p$  state for small values of  $\delta$  since the effective potential (3) becomes closer to the original Hulthén potential (1) and for small  $l$  the contribution of this angular momentum term in potential is also small. It is found that if all the parameters of potential remain purely real, then all bound energies  $E_n$  with  $n \geq 0$  will have negative energy spectrum [29]. We also point out that this approximate results obtained for the Hulthén potential may have some interesting applications in the study of different quantum mechanical systems and atomic physics.

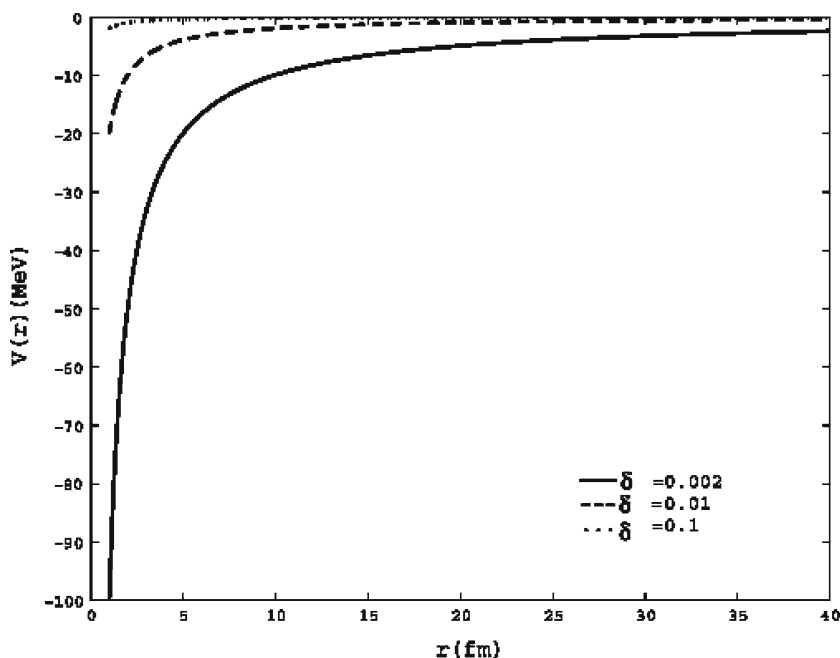


Figure 2. Variation of the Hulthén potential as a function  $r$ . The curves are shown for  $s$ -state with various values of the screening parameter  $\delta = 0.002, 0.01$ , and  $0.1$ .



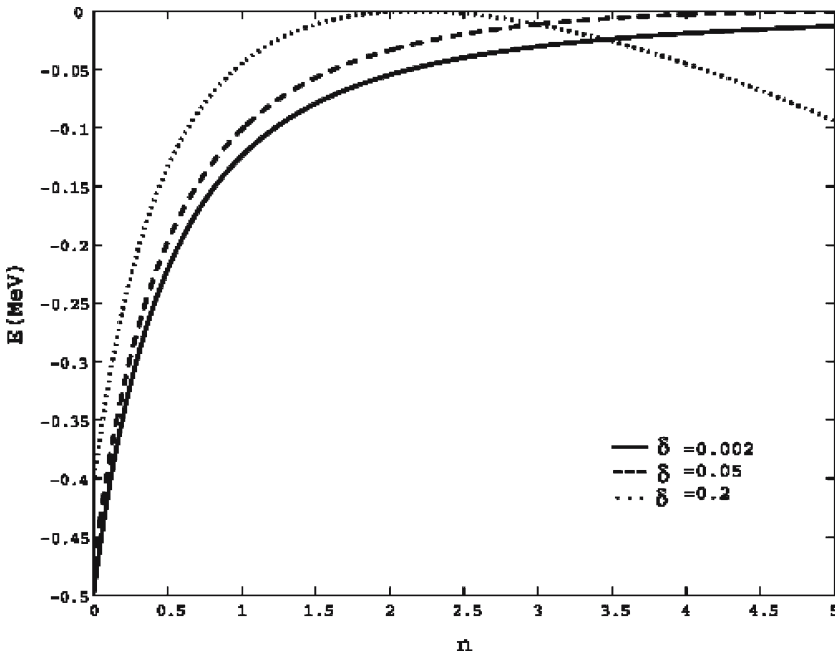


Figure 3. The variation of the energy eigenvalues with respect to the quantum number  $n$ . The curves shown are for  $s$ -state with various values of the screening parameters  $\delta = 0.002, 0.05,$  and  $0.2$ .

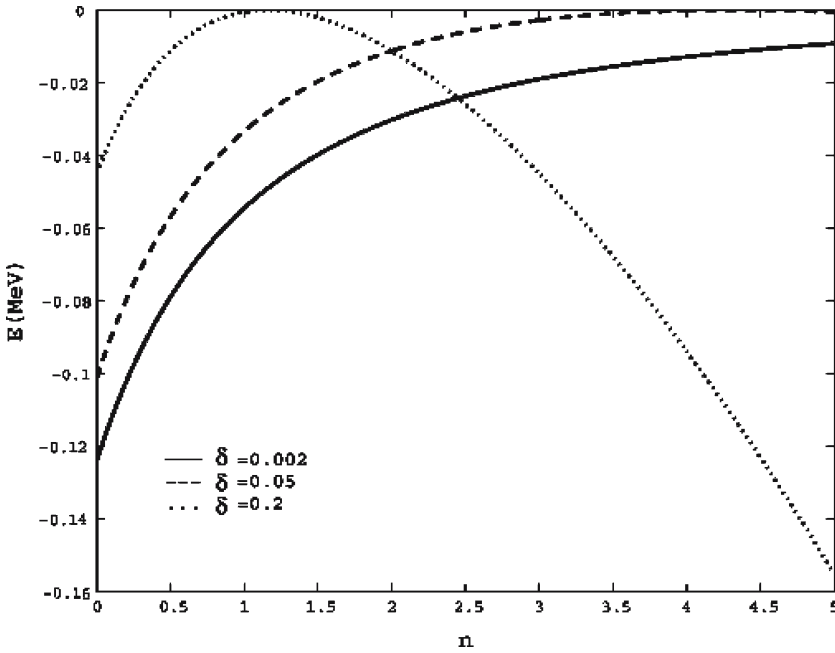


Figure 4. The variation of the energy eigenvalues with respect to the quantum number  $n$ . The curves shown are for  $p$ -state with the same screening parameters as in figure 3.

Table 1  
The  $s$ -states energy eigenvalues of the Hulthén potential for several values of screening parameter  $\delta$ .

$n$	$-E_n$ [11]	$-\bar{E}_n$ [30]	$-E_{\text{exact}}$	Our work
$\delta = 0.002$				
1	0.4990005	0.4990005	0.4990005	0.4990005
2	0.1240020	0.1240020	0.1240020	0.1240020
3	0.0545601	0.0545601	0.0545601	0.0545601
4	0.0302580	0.0302580	0.0302580	0.0302580
5				0.0190125
$\delta = 0.01$				
1	0.4950125	0.4950125	0.4950125	0.4950125
2	0.1200500	0.1200500	0.1200500	0.1200500
3	0.0506681	0.0506681	0.0506681	0.0506681
4	0.0264501	0.0264500	0.0264500	0.0264500
5	0.0153128	0.0153125	0.0153125	0.0153125
$\delta = 0.05$				
1	0.4753125	0.4753125	0.4753125	0.4753125
2	0.1012503	0.1012500	0.1012500	0.1012500
3	0.0333746	0.0333681	0.0333681	0.0333681
4	0.0113035	0.0112500	0.0112500	0.0112500
5		0.0028125		0.0028125
$\delta = 0.2$				
1	0.4049962	0.4050000	0.4050000	0.4050000
2	0.0450856	0.0450000	0.0450000	0.0450000
3		0.0005556		0.0005556
4		0.0112500		0.0112500

Table 2  
Energy eigenvalues as a function of the screening parameter  $\delta$  for the states  $2p$  and  $3d$ .

State	$\delta$	$-E_{n,l}$ [3] <sup>a</sup>	$-E_{n,l}$ [27] <sup>b</sup>	Our work <sup>c</sup>
$2p$	0.025	0.112760	0.1127605	0.1128125
	0.050	0.101042	0.1010425	0.1012500
	0.075	0.089845	0.0898478	0.0903125
	0.100	0.079170	0.0791794	0.0800000
	0.150	0.059495	0.0594415	0.0612500
	0.200	0.041792	0.0418860	0.0450000
$3d$	0.025	0.043601	0.0437069	0.0437590
	0.050	0.032748	0.3316450	0.0333681
	0.075	0.023010	0.0239397	0.0243837
	0.100	0.014433	0.0160537	0.0168056

<sup>a</sup>Variational method.

<sup>b</sup>Numerical integration.

<sup>c</sup>The small difference in results is because the potential form used by [27] for  $l \neq 0$  part is different than our form.

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